國立臺灣大學109學年度轉學生招生考試試題

題號： 19
科目：微積分(B)

題號： 19
共 2 頁之第 1 頁

※ 注意：請於試卷上「非選擇題作答區」標明大題及小題題號，並依序作答。

Any device with computer algebra system is prohibited during the exam.

PART 1 : Fill in the blanks.

- Only answers will be graded.
- Each answer must be clearly labeled on the answer sheet.
- 4 points are assigned to each blank.

1. (a) $\lim_{x\to\infty}(1+\frac{3}{x})^{[x]} =$ ___(1)___ , where $[x]$ is the greatest integer which is less than or equal to x.

   (b) $\lim_{x\to 0}(\cos ax)^{\frac{1}{1-\cos bx}} =$ ___(2)___ , where a, b are constants and $ab \neq 0$.

   (c) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2n\sqrt{1-(\frac{i}{2n})^2}} =$ ___(3)___ .

命中&相似題目 ： 講義 p.6-22 Ex23; p.6-33 Exercise 33 (相似度 90%)

Ex 23.

Evaluate $\lim_{n\to\infty}\sin\frac{\pi}{n}\cdot\sum_{k=1}^{n}\frac{1}{2+\cos\frac{k\pi}{n}}$.

Solution. $\frac{\pi}{\sqrt{3}}$

Exercise 23.

Compute $\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\frac{4k}{n}}\cdot\frac{4}{n}$

Ans.

[觀點 1]： 令 $f(x)=\sqrt{x},\ x_k=\frac{4k}{n},\ k=1,2,\cdots,n,\ \Delta x=\frac{4}{n}$. 則

$$\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\frac{4k}{n}}\cdot\frac{4}{n}=\lim_{n\to\infty}\sum_{k=1}^{n}f(x_k)\cdot\Delta x=\int_0^4\sqrt{x}\,dx=\frac{2}{3}x^{3/2}\Big|_0^4=\frac{16}{3}. \quad \#$$

[觀點 2]： 令 $f(x)=\sqrt{4x},\ x_k=\frac{k}{n},\ k=1,2,\cdots,n,\ \Delta x=\frac{1}{n}$. 則

$$\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\frac{4k}{n}}\cdot\frac{4}{n}=4\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\frac{k}{n}}\cdot 4\frac{1}{n}=\lim_{n\to\infty}\sum_{k=1}^{n}4f(x_k)\cdot\Delta x=\int_0^1\sqrt{x}\,4dx=4\cdot\frac{2}{3}x^{3/2}\Big|_0^1=8\frac{1}{3}.\ \#$$

6. (a) $\iint_D y\,dA =$ __(15)__, where D is the region in the first quadrant that lies between the circles $x^2+y^2=4$ and $(x-1)^2+y^2=1$.

命中&相似題目 ： 講義 p.12-20 Exercise3 (相似度 99%)

Exercise 3.

Evaluate $\iint_D x\,dA$, , where D is the region in the first quadrant that lies between the circles $x^2+y^2=4$ and $x^2+y^2=2x$

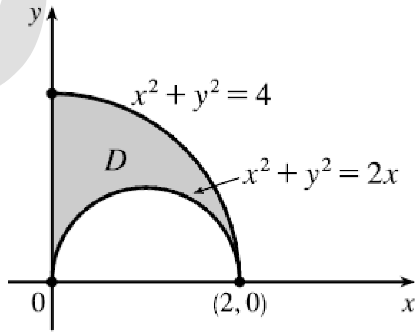
Ans.

$$\iint_D x\,dA = \iint_{\substack{x^2+y^2\le 4\\ x,y\ge 0}} x\,dA - \iint_{\substack{(x-1)^2+y^2\le 1\\ y\ge 0}} x\,dA$$

$$= \int_0^{\pi/2}\int_0^2 r^2\cos\theta\,dr\,d\theta - \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\frac{1}{3}(8\cos\theta)\,d\theta - \int_0^{\pi/2}\frac{1}{3}(8\cos^4\theta)\,d\theta$$

$$= \frac{8}{3}-\frac{\pi}{2}.$$ #



5. Let the region R be enclosed by the curves $y = x^2$ and $y = 2 - x^2$. Find the volume of the solid obtained by rotating the region R about $x = 1$.

命中&相似題目 ： 講義 p.7-20 Homework7.2 #5 (相似度 100%)

Homework 7.2

1-6, Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

1. $y = x+1, y = 0, x = 0, x = 2$, about the x-axis.

2. $x = y - y^2, x = 0$, about the y-axis

3. $y = x^2, x = y^2$, about $y = 1$.

4. $y = \frac{1}{x}, y = 0, x = 1, x = 2$, about y -axis.

5. $y = x^2, y = 2 - x^2$, about $x = 1$.

7. Evaluate $\iint_R \cos(\frac{y-x}{y+x})dA$ where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,2)$ and $(0,1)$.

命中&相似題目 ： 講義 p.12-22 Ex16. (相似度 100%)

Ex 16.

Evaluate $\iint_R \cos\left(\frac{y-x}{y+x}\right) dA$, where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,2)$ and $(0,1)$.

Solution. $\frac{3}{2}\sin 1$

8. Let $f(x,y) = x^4 + y^4 - 4xy + 1$. Find local maxima, local minima, and saddle points of $f(x,y)$.

命中&相似題目 ： 講義 p.11-35 Exercise9. (相似度 99%)

> Exercise 9.
>
> Find all the local maxima, local minima, and saddle points of the function $f(x,y) = 4xy - x^4 - y^4$.

Ans.

$$\begin{cases} f_x = 4y - 4x^3 = 0 \\ f_y = 4x - 4y^3 = 0 \end{cases} \Rightarrow \begin{cases} y = x^3 \\ x = y^3 \end{cases} \Rightarrow x = x^9 \Rightarrow x\left(x^8 - 1\right) = 0 \Rightarrow x = 0, \pm 1.$$

$\therefore (x,y) = (0,0),\ (1,1),\ (-1,-1)$ 為臨界點.

$f_{xx} = -12x^2,\ f_{yy} = -12y^2,\ f_{xy} = 4,\ \Delta(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = 144x^2y^2 - 4$

$\because \Delta(0,0) = -4 < 0 \quad \therefore (0,0)$ 為鞍點.

$\because \Delta(1,1) = 140 > 0$ 且 $f_{xx}(1,1) = -12 < 0 \quad \therefore f(1,1) = 4 - 1 - 1$ 為局部最大值.

$\because \Delta(-1,-1) = 140 > 0$ 且 $f_{xx}(-1,-1) = -12 < 0 \quad \therefore f(-1,-1) = 4 - 1 - 1$ 為局部最大值. #

9. Find the absolute maximum value and absolute minimum value of $f(x,y) = x^4 + y^4 - 4xy + 1$ on the disk $x^2 + y^2 \le 1$.

命中&相似題目 ： 講義 p.11-28 Ex22. (相似度 90%)

> Ex 22.
> Find the extreme values of the function $f(x,y) = x^2 + 2y^2$ on the region $x^2 + y^2 \le 1$.

Solution. Max.=2, min.=0

10. Solve the differential equation $xy' = y + x^2 \sin x$ with $y(\pi) = 2\pi$.

命中&相似題目 ： 講義 p.9-5 Ex5. (相似度 90%)

> Ex 5.
> Find the solution of the initial-value problem $x^2 y' + xy = 1,\ x > 0,\ y(1) = 2$.

Solution. $y(x) = \dfrac{\ln x + 2}{x}$

1. $\lim_{x\to-\infty}\left(3x+\sqrt{9x^2-x}\right)=?$

命中&相似題目：微積分學習要訣 P.1-14 例 9

1-14 微積分學習要訣

[解]注意負號！令 $x=-t$ 代入得

$$\text{原式}=\lim_{t\to\infty}\frac{\sqrt{9(-t)^2+1}}{-t+4}=\lim_{t\to\infty}\frac{\sqrt{9t^2+1}}{-t+4}=\lim_{t\to\infty}\frac{\sqrt{9+\frac{1}{t^2}}}{-1+\frac{4}{t}}=\frac{\sqrt{9}}{-1}=-3\text{。}$$

* * *

類 求 $\lim_{x\to-\infty}\frac{\sqrt{16x^2+7}}{-5x+8}=?$

答：令 $x=-t$ 代入得原式 $=\lim_{t\to\infty}\frac{\sqrt{16(-t)^2+7}}{5t+8}=\lim_{t\to\infty}\frac{\sqrt{16t^2+7}}{5t+8}=\frac{4}{5}$。

說例 9 注意題	求 $\lim_{x\to-\infty}\left(x+\sqrt{x^2+2x}\right)=?$ （政大轉）

[解]令 $x=-t$ 代入得

$$\text{原式}=\lim_{t\to\infty}(-t+\sqrt{t^2-2t})=\lim_{t\to\infty}\frac{(\sqrt{t^2-2t}-t)(\sqrt{t^2-2t}+t)}{\sqrt{t^2-2t}+t}$$

$$=\lim_{t\to\infty}\frac{t^2-2t-t^2}{\sqrt{t^2-2t}+t}=\frac{-2}{1+1}=-1\text{。}$$

* * *

類 求 $\lim_{x\to-\infty}\left(\sqrt{4x^2+5x+2}+2x\right)=?$

答：令 $x=-t$ 代入得

$$\text{原式}=\lim_{t\to\infty}(\sqrt{4t^2-5t+2}-2t)=\lim_{t\to\infty}\frac{(4t^2-5t+2)-4t^2}{\sqrt{4t^2-5t+2}+2t}$$

$$=\lim_{t\to\infty}\frac{-5t+2}{\sqrt{4t^2-5t+2}+2t}=-\frac{5}{4}\text{。}$$

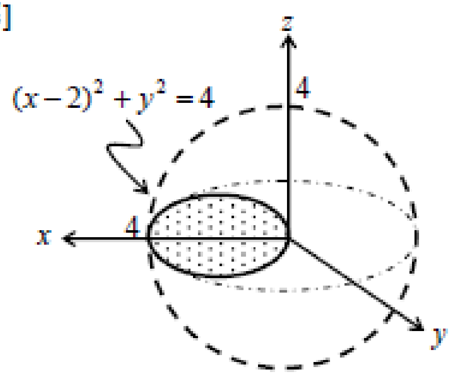
說例 10 觀念題	設 $f(x)=\frac{a\sqrt{x^2+5}-b}{x-2}$，若 $\lim_{x\to\infty}f(x)=1$，且 $\lim_{x\to2}f(x)$ 存在，(1)求 a、$b=?$ (2)求 $\lim_{x\to2}f(x)=?$

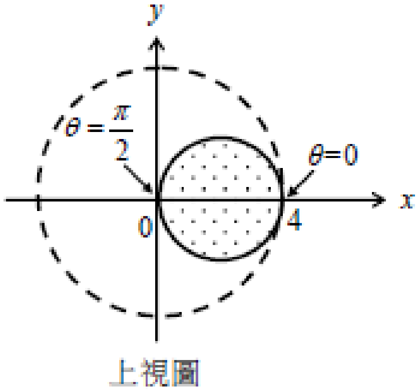
8. Find the volume determined by $x^2+y^2+z^2\le 1$ and $x^2+y^2\le y$.

命中&相似題目：微積分學習要訣 P.9-151 類題

第九章 線積分與重積分 *9-151*

[解]





$$V=\iiint_V dzdxdy=\iint_R\int_{z=-\sqrt{16-(x^2+y^2)}}^{z=\sqrt{16-(x^2+y^2)}}dzdxdy=\iint_{(x-2)^2+y^2\le 2^2}2\sqrt{16-(x^2+y^2)}dxdy$$

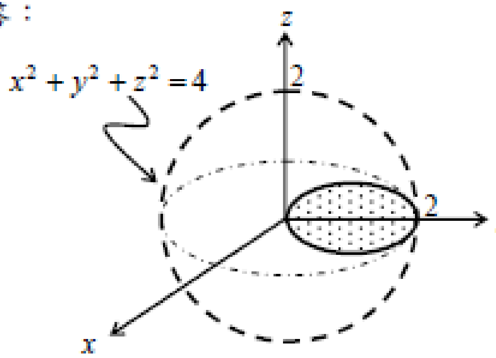
$$=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{r=0}^{r=4\cos\theta}\sqrt{16-r^2}\,rdrd\theta=2\cdot 2\int_0^{\frac{\pi}{2}}\int_{r=0}^{r=4\cos\theta}\sqrt{16-r^2}\,rdrd\theta\text{（依對稱性）}$$

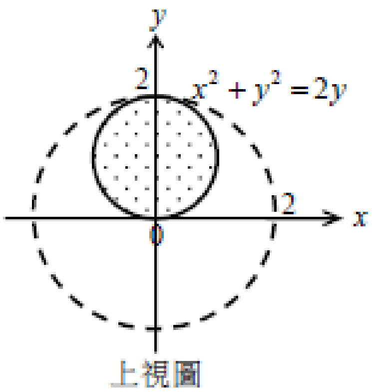
$$=4\int_0^{\frac{\pi}{2}}\left[-\frac{1}{3}(16-r^2)^{3/2}\right]_0^{4\cos\theta}d\theta=\frac{256}{3}\int_0^{\frac{\pi}{2}}(1-\sin^3\theta)d\theta$$

$$=\frac{256}{3}\left[\frac{\pi}{2}-\frac{2}{3\cdot 1}\right]=\frac{256}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right)\text{。}$$

類 求球体 $x^2+y^2+z^2\le 4$ 被圓柱 $r\le 2\sin\theta$ 切除之体積？(成大轉)

答：





$$\therefore V=\iiint_V dzdxdy=\iint_R\int_{z=-\sqrt{4-(x^2+y^2)}}^{z=\sqrt{4-(x^2+y^2)}}dzdxdy=\iint_{r\le 2\sin\theta}2\sqrt{4-(x^2+y^2)}dxdy$$