國立臺灣大學109學年度轉學生招生考試試題

題號： 19
科目：微積分(B)

題號： 19
共 2 頁之第 1 頁

※ 注意：請於試卷上「非選擇題作答區」標明大題及小題題號，並依序作答。

Any device with computer algebra system is prohibited during the exam.

PART 1 : Fill in the blanks.

- Only answers will be graded.
- Each answer must be clearly labeled on the answer sheet.
- 4 points are assigned to each blank.

1. (a) $\lim_{x\to\infty}(1+\frac{3}{x})^{[x]} =$ __(1)__ , where $[x]$ is the greatest integer which is less than or equal to x.

   (b) $\lim_{x\to 0}(\cos ax)^{\frac{1}{1-\cos bx}} =$ __(2)__ , where a, b are constants and $ab \neq 0$.

   (c) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2n\sqrt{1-(\frac{i}{2n})^2}} =$ __(3)__ .

命中&相似題目 ： 講義 p.6-22 Ex23; p.6-33 Exercise 33 (相似度 90%)

Ex 23.

Evaluate $\lim_{n\to\infty}\sin\frac{\pi}{n}\cdot\sum_{k=1}^{n}\frac{1}{2+\cos\frac{k\pi}{n}}$.

Solution. $\frac{\pi}{\sqrt{3}}$

Exercise 23.

Compute $\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\frac{4k}{n}}\cdot\frac{4}{n}$

Ans.

[觀點 1]：令 $f(x)=\sqrt{x},\ x_k=\frac{4k}{n},\ k=1,2,\cdots,n,\ \Delta x=\frac{4}{n}$. 則

$$\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\frac{4k}{n}}\cdot\frac{4}{n}=\lim_{n\to\infty}\sum_{k=1}^{n}f(x_k)\cdot\Delta x=\int_0^4\sqrt{x}\,dx=\frac{2}{3}x^{3/2}\Big|_0^4=\frac{16}{3}.\quad \#$$

[觀點 2]：令 $f(x)=\sqrt{4x},\ x_k=\frac{k}{n},\ k=1,2,\cdots,n,\ \Delta x=\frac{1}{n}$. 則

$$\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\frac{4k}{n}}\cdot\frac{4}{n}=4\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\frac{k}{n}}\cdot 4\cdot\frac{1}{n}=\lim_{n\to\infty}4\sum_{k=1}^{n}f(x_k)\cdot\Delta x=\int_0^1\sqrt{x}\,4dx=4\cdot\frac{2}{3}x^{3/2}\Big|_0^1=8\cdot\frac{1}{3}.\ \#$$

6. (a) $\iint_D y\,dA =$ __(15)__, where D is the region in the first quadrant that lies between the circles $x^2+y^2=4$ and $(x-1)^2+y^2=1$.

命中&相似題目 ： 講義 p.12-20 Exercise3 (相似度 99%)

Exercise 3.

Evaluate $\iint_D x\,dA$, , where D is the region in the first quadrant that lies between the circles $x^2+y^2=4$ and $x^2+y^2=2x$

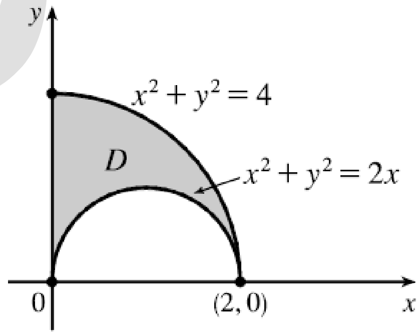
Ans.

$$\iint_D x\,dA = \iint_{\substack{x^2+y^2\le 4\\ x,y\ge 0}} x\,dA - \iint_{\substack{(x-1)^2+y^2\le 1\\ y\ge 0}} x\,dA$$

$$= \int_0^{\pi/2}\int_0^2 r^2\cos\theta\,dr\,d\theta - \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\frac{1}{3}(8\cos\theta)\,d\theta - \int_0^{\pi/2}\frac{1}{3}(8\cos^4\theta)\,d\theta$$

$$= \frac{8}{3}-\frac{\pi}{2}. \quad \#$$



5. Let the region R be enclosed by the curves $y = x^2$ and $y = 2 - x^2$. Find the volume of the solid obtained by rotating the region R about $x = 1$.

命中&相似題目 ： 講義 p.7-20 Homework7.2 #5 (相似度 100%)

Homework 7.2

1-6, Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

1. $y = x+1, y = 0, x = 0, x = 2$, about the x-axis.
2. $x = y - y^2, x = 0$, about the y-axis
3. $y = x^2, x = y^2$, about $y = 1$.
4. $y = \frac{1}{x}, y = 0, x = 1, x = 2$, about y-axis.
5. $y = x^2, y = 2 - x^2$, about $x = 1$.

7. Evaluate $\iint_R \cos(\frac{y-x}{y+x})dA$ where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,2)$ and $(0,1)$.

命中&相似題目 ： 講義 p.12-22 Ex16. (相似度 100%)

Ex 16.

Evaluate $\iint_R \cos\left(\frac{y-x}{y+x}\right) dA$, where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,2)$ and $(0,1)$.

Solution. $\frac{3}{2}\sin 1$

8. Let $f(x,y)=x^4+y^4-4xy+1$. Find local maxima, local minima, and saddle points of $f(x,y)$.

命中&相似題目 ： 講義 p.11-35 Exercise9. (相似度 99%)

<u>Exercise 9.</u>

Find all the local maxima, local minima, and saddle points of the function $f(x,y)=4xy-x^4-y^4$.

<u>Ans.</u>

$$\begin{cases} f_x=4y-4x^3=0 \\ f_y=4x-4y^3=0 \end{cases} \Rightarrow \begin{cases} y=x^3 \\ x=y^3 \end{cases} \Rightarrow x=x^9 \Rightarrow x\left(x^8-1\right)=0 \Rightarrow x=0,\pm 1.$$

$\therefore (x,y)=(0,0),\ (1,1),\ (-1,-1)$ 為臨界點.

$f_{xx}=-12x^2,\ f_{yy}=-12y^2,\ f_{xy}=4,\ \Delta(x,y)=f_{xx}f_{yy}-(f_{xy})^2=144x^2y^2-4$

$\because \Delta(0,0)=-4<0$ $\therefore (0,0)$ 為鞍點.

$\because \Delta(1,1)=140>0$ 且 $f_{xx}(1,1)=-12<0$ $\therefore f(1,1)=4-1-1$ 為局部最大值.

$\because \Delta(-1,-1)=140>0$ 且 $f_{xx}(-1,-1)=-12<0$ $\therefore f(-1,-1)=4-1-1$ 為局部最大值. #

9. Find the absolute maximum value and absolute minimum value of $f(x,y)=x^4+y^4-4xy+1$ on the disk $x^2+y^2\le 1$.

命中&相似題目 ： 講義 p.11-28 Ex22. (相似度 90%)

<u>Ex 22.</u>

Find the extreme values of the function $f(x,y)=x^2+2y^2$ on the region $x^2+y^2\le 1$.

<u>Solution.</u> Max.=2, min.=0

10. Solve the differential equation $xy'=y+x^2\sin x$ with $y(\pi)=2\pi$.

命中&相似題目 ： 講義 p.9-5 Ex5. (相似度 90%)

<u>Ex 5.</u>

Find the solution of the initial-value problem $x^2y'+xy=1,\ x>0,\ y(1)=2$.

<u>Solution.</u> $y(x)=\dfrac{\ln x+2}{x}$

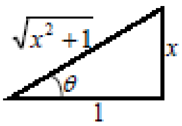
9. $\displaystyle\int_0^1 \frac{1}{(x^2+1)^2}\,dx =$ ____(10)____.

命中&相似題目：微積分學習要訣 P.4-55

$$=\frac{1}{4a}\left[-\frac{1}{2}\ln(x^2+2ax+2a^2)+\tan^{-1}(\frac{x+a}{a})\right]$$
$$+\frac{1}{4a}\left[\frac{1}{2}\ln(x^2-2ax+2a^2)+\tan^{-1}(\frac{x-a}{a})\right]+C \text{。}$$

說例 9 漂亮題	求 $\displaystyle\int\frac{1}{(x^2+1)^2}dx=?$

[解]令 $x=\tan\theta,\ dx=\sec^2\theta\, d\theta$



$$\text{原式}=\int\frac{\sec^2\theta}{(\tan^2\theta+1)^2}d\theta=\int\frac{\sec^2\theta}{\sec^4\theta}d\theta$$
$$=\int\cos^2\theta\, d\theta=\int\frac{1+\cos2\theta}{2}d\theta$$
$$=\frac{\theta}{2}+\frac{1}{4}\sin2\theta+C$$
$$=\frac{1}{2}\tan^{-1}x+\frac{1}{4}\cdot2\cdot\frac{x}{\sqrt{x^2+1}}\cdot\frac{1}{\sqrt{x^2+1}}+C$$
$$=\frac{1}{2}\tan^{-1}x+\frac{x}{2(x^2+1)}+C \text{。}$$

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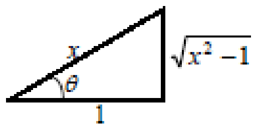
類 求 $\displaystyle\int\frac{1}{(x^2-1)^2}dx=?$

答：令 $x=\sec\theta,\ dx=\sec\theta\tan\theta\, d\theta$



$$\text{原式}=\int\frac{\sec\theta\tan\theta}{(\tan^2\theta)^2}d\theta=\int\cot^3\theta\sec\theta\, d\theta$$
$$=\int\cot^2\theta\csc\theta\, d\theta=\int(\csc^2\theta-1)\csc\theta\, d\theta$$
$$=\int\csc^3\theta\, d\theta-\int\csc\theta\, d\theta$$
$$=-\frac{1}{2}\left[\csc\theta\cot\theta+\ln\left|\csc\theta+\cot\theta\right|\right]+\ln\left|\csc\theta+\cot\theta\right|+C$$
$$=-\frac{1}{2}\csc\theta\cot\theta+\frac{1}{2}\ln\left|\csc\theta+\cot\theta\right|+C$$
$$=-\frac{1}{2}\frac{x}{x^2-1}+\frac{1}{2}\ln\left|\sqrt{\frac{x+1}{x-1}}\right|+C \text{。}$$

13. If a and b are positive constants and if $\max\{p,q\}$ denotes the maximum between the numbers p and q, the iterated integral $\int_0^a \int_0^b e^{\max\{b^2x^2,\, a^2y^2\}}\, dy\, dx =$ ____(17)____.

命中&相似題目：微積分考前總攻略 P.9-77

微積分考前總攻略 9-77

$$=\int_0^1\int_0^1 \frac{1}{yz}\left[yz-\frac{1}{2}y^2z^2+\frac{1}{3}y^3z^3-\cdots\right]dydz$$

$$=\int_0^1\int_0^1\left[1-\frac{1}{2}yz+\frac{1}{3}y^2z^2-\cdots\right]dydz$$

$$=\int_0^1\left[y-\frac{1}{4}y^2z+\frac{1}{9}y^3z^2-\cdots\right]_{y=0}^{y=1}dz$$

$$=\int_0^1\left[1-\frac{1}{4}z+\frac{1}{9}z^2-\cdots\right]dz=\left[z-\frac{1}{8}z^2+\frac{1}{27}z^3-\cdots\right]_0^1$$

$$=1-\frac{1}{8}+\frac{1}{27}-\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}\text{ 。}$$

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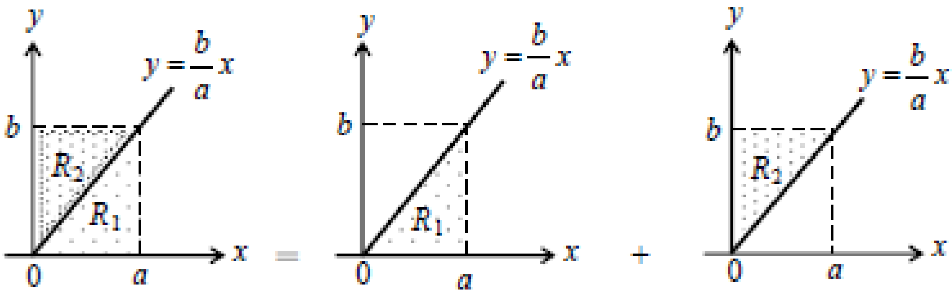
試題【中央研】

求 $\int_0^b\int_0^a e^{\max(b^2x^2,a^2y^2)}dxdy=?$

答

將積分區域分割如下：



$$\text{原式}=\int_0^a\int_0^{\frac{b}{a}x}e^{b^2x^2}dydx+\int_0^b\int_0^{\frac{a}{b}y}e^{a^2y^2}dxdy$$

$$=\int_0^a\left[ye^{b^2x^2}\right]_{y=0}^{y=\frac{b}{a}x}dx+\int_0^b\left[xe^{a^2y^2}\right]_{x=0}^{x=\frac{a}{b}y}dy=\int_0^a\frac{b}{a}xe^{b^2x^2}dx+\int_0^b\frac{a}{b}ye^{a^2y^2}dy$$

$$=\left[\frac{1}{2ab}e^{b^2x^2}\right]_0^a+\left[\frac{1}{2ab}e^{a^2y^2}\right]_0^b=\frac{1}{2ab}(e^{a^2b^2}-1)+\frac{1}{2ab}(e^{a^2b^2}-1)$$